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Opening up extra dimensions at ultra-large scales

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Abstract

The standard picture of viable higher-dimensional theories is that direct manifestations of extra dimensions occur at short distances only, whereas long-distance physics is described by effective four-dimensional theories. We show that this is not necessarily true in models with infinite extra dimensions. As an example, we consider a five-dimensional scenario with three 3-branes in which gravity is five-dimensional both at short *and* very long distance scales, with conventional four-dimensional gravity operating at intermediate length scales. A phenomenologically acceptable range of validity of four-dimensional gravity extending from microscopic to cosmological scales is obtained without strong fine-tuning of parameters.

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Our world is often thought to have more than four fundamental space-time dimensions. This point of view is strongly supported by string/M-theory, and higher dimensional theories are currently being developed in various directions. The standard lore is that in phenomenologically viable models, extra dimensions open up at short distances only, whereas above a certain length scale, and all the way up to infinite distances, physics is described by effective four-dimensional theories.

In this letter we show that the latter picture is not universal: there are models in which extra dimensions open up both at short *and very long* distances. Namely, gravity may become higher-dimensional in both of these extremes. At intermediate length scales, physics is described by conventional four-dimensional laws, so the phenomenology of these models is still acceptable.

The starting point for our discussion is the observation [1] that extra dimensions may be infinite, with the usual matter residing on a 3-brane embedded in higher-dimensional space. In the original Randall–Sundrum (RS) model [1], gravity is effectively four-dimensional at large scales due to the existence of a graviton bound state localized near the 3-brane in five dimensions. We point out that in other five-dimensional models with an infinite extra dimension (in particular, in the model introduced in Ref. [2]), localization of the graviton may be incomplete, and it is this property that leads to the restoration of the five-dimensional form of gravity at very long distances. The latter feature is manifest both in the case of static sources, where the four-dimensional Newton's gravity law changes to its five-dimensional counterpart above a certain length scale, and in dynamics of gravitational waves which disappear into the fifth dimension thus violating four-dimensional energy conservation, after travelling long enough distance. It is likely that these phenomena are not peculiar to five-dimensional theories and occur in a number of models with more than one extra dimension.

It has been found recently [3] that exotic large-distance effects may also appear in models with compact extra dimensions, due to the possible presence of very light Kaluza–Klein states. This interesting scenario is, however, considerably different from ours: in compact models, extra dimensions show up at large distances somewhat indirectly, through the spectrum of the corresponding four-dimensional effective theory: four-dimensional energy is conserved, and so on. In our case, physics at ultra-large distances in intrinsically five-dimensional.

As our concrete example, let us consider a five-dimensional model of Ref. [2]. The model contains one brane with tension $\sigma > 0$ and two branes with equal tensions $-\sigma/2$ placed at equal distances to the right and to the left of the positive tension brane in the fifth direction. The two negative tension branes are introduced for simplicity, to have Z_2 symmetry, $z \to -z$, in analogy to RS model (hereafter z denotes the fifth coordinate). We assume that conventional matter resides on the positive tension brane, and in what follows we will be interested in gravitational interactions of this matter. The Z_2 symmetry enables us to consider explicitly only the region to the right of the positive tension brane.

The physical set-up (for $z > 0$) is as follows: The bulk cosmological constant between the branes, Λ, is negative as in the RS model, however, in contrast to that model, is *zero* to the right of the negative tension brane. With appropriately tuned Λ, there exists a solution to the five-dimensional Einstein equations for which both positive and negative tension branes are at rest at $z = 0$ and $z = z_c$ respectively, z_c being an arbitrary constant. The metric of this solution is

$$ds^2 = a^2(z)\eta_{\mu\nu}dx^\mu dx^\nu - dz^2 \tag{1}$$

where

$$a(z) = \begin{cases} e^{-kz} & 0 < z < z_c \\ e^{-kz_c} \equiv a_- & z > z_c \end{cases} \tag{2}$$

The constant k is related to σ and Λ as follows: $\sigma = \frac{3k}{4\pi G_5}$, $\Lambda = -\sigma k$, where G_5 is the five-dimensional Newton constant. The four-dimensional hypersurfaces $z = const.$ are flat, the five-dimensional space-time is flat to the right of the negative-tension brane and anti-de Sitter between the branes. The spacetime to the left of the positive tension brane is of course a mirror image of this set-up.

This background has two length scales, k^{-1} and $\zeta_c \equiv k^{-1}e^{kz_c}$. We will consider the case of large enough z_c, in which the two scales are well separated, $\zeta_c \gg k^{-1}$. We will see that gravity in this model is effectively four-dimensional at distances r belonging to the interval $k^{-1} \ll r \ll \zeta_c(k\zeta_c)^2$, and is five-dimensional *both* at short distances, $r \ll k^{-1}$ (this situation is exactly the same as in RS model), and at long distances, $r \gg \zeta_c(k\zeta_c)^2$. In the latter régime of very long distances the five-dimensional gravitational constant gets effectively renormalized and no longer coincides with G_5.

To find the gravity law experienced by matter residing on the positive tension brane, let us study gravitational perturbations about the background metric (1). We will work in the Gaussian Normal (GN) gauge, $g_{zz} = -1$, $g_{z\mu} = 0$. The linearized theory is described by the metric

$$ds^2 = a^2(z)\eta_{\mu\nu}dx^\mu dx^\nu + h_{\mu\nu}(x, z)dx^\mu dx^\nu - dz^2 \tag{3}$$

There are two types of linearized excitations in this model. One of them is a four-dimensional scalar — the radion — that corresponds to the relative motion of the branes (see, e.g., Refs. [4,5,2]) The wave function of the radion is localized between the branes, and its interactions with matter on the positive tension brane result in a scalar force of the Brans–Dicke type. If the distance between the branes is not stabilized by one or another mechanism, the radion has zero four-dimensional mass and gives rise to the usual $1/r$ potential in an effective four-dimensional theory. In the particular model under discussion, the radion excitation has been studied in Ref. [2]. It is conceivable that the distance between the branes may be stabilized (cf. Ref. [6]); in which case the interactions due to the radion switch off at large distances.

In this letter we are interested in other types of excitation that leave the branes at rest, namely the five-dimensional gravitons. When the radion is disregarded, there exists a frame which is GN with respect to both branes simultaneously. In this frame, the transverse-traceless gauge can be chosen, $h^\mu_\mu = 0$, $h^\nu_{\mu,\nu} = 0$, and the linearized Einstein equations take one and the same simple form for all components of $h_{\mu\nu}$,

$$\begin{cases} h'' - 4k^2 h - \frac{1}{a^2}\Box^{(4)}h = 0 & 0 < z < z_c \\ h'' - \frac{1}{a_-^2}\Box^{(4)}h = 0 & z > z_c \end{cases} \tag{4}$$

The Israel junction conditions on the branes are

$$\begin{cases} h' + 2kh = 0 & \text{at } z = 0 \\ [h'] - 2kh = 0 & \text{at } z = z_c \end{cases} \tag{5}$$

where $[h']$ is the discontinuity of the z-derivative of the metric perturbation at z_c, and four-dimensional indices are omitted. A general perturbation is a superposition of modes, $h = \psi(z)e^{ip_\mu x^\mu}$ with $p^2 = m^2$, where ψ obeys the following set of equations in the bulk,

$$\begin{cases} \psi'' - 4k^2\psi + \frac{m^2}{a^2}\psi = 0 & 0 < z < z_c \\ \psi'' + \frac{m^2}{a_-^2}\psi = 0 & z > z_c \end{cases} \tag{6}$$

with the junction conditions (5) (replacing h by ψ). It is straightforward to check that there are no negative modes, i.e., normalizable solutions to these equations with $m^2 < 0$. There are no normalizable solutions with $m^2 \geq 0$ either, so the spectrum is continuous, beginning at $m^2 = 0$. To write the modes explicitly, it is convenient to introduce a new coordinate between the branes, $\zeta = \frac{1}{k}e^{kz}$, in terms of which the background metric is conformally flat. Then the modes have the following form,

$$\psi_m = \begin{cases} C_m \left[N_1\left(\frac{m}{k}\right) J_2(m\zeta) - J_1\left(\frac{m}{k}\right) N_2(m\zeta) \right] & 0 < z < z_c \\ A_m \cos\left(\frac{m}{a_-}(z - z_c)\right) + B_m \sin\left(\frac{m}{a_-}(z - z_c)\right) & z > z_c \end{cases} \tag{7}$$

where N and J are the Bessel functions. The constants A_m, B_m and C_m obey two relations due to the junction conditions at the negative tension brane. Explicitly,

$$A_m = C_m \left[N_1\left(\frac{m}{k}\right) J_2(m\zeta_c) - J_1\left(\frac{m}{k}\right) N_2(m\zeta_c) \right] \tag{8a}$$

$$B_m = C_m \left[N_1\left(\frac{m}{k}\right) J_1(m\zeta_c) - J_1\left(\frac{m}{k}\right) N_1(m\zeta_c) \right] \tag{8b}$$

The remaining overall constant C_m is obtained from the normalization condition. The latter is determined by the explicit form of Eq. (6) and reads

$$\int \psi_m^*(z)\psi_{m'}(z)\frac{dz}{a^2(z)} = \delta(m - m') \tag{9}$$

One makes use of the asymptotic behaviour of ψ_m at $z \to \infty$ and finds

$$\frac{\pi}{a_-}(|A_m|^2 + |B_m|^2) = 1 \tag{10}$$

which fixes C_m from (8a) and (8b).

It is instructive to consider two limiting cases. At $m\zeta_c \gg 1$ we obtain by making use of the asymptotics of the Bessel functions,

$$C_m^2 = \frac{m}{2k}\left[J_1^2\left(\frac{m}{k}\right) + N_1^2\left(\frac{m}{k}\right) \right]^{-1} \tag{11}$$

which coincides, as one might expect, with the normalization factor for the massive modes in RS model. In the opposite case $m\zeta_c \ll 1$ (notice that this automatically implies $m/k \ll 1$), the expansion of the Bessel functions in Eqs. (8a) and (8b) yields

$$C_m^2 = \frac{\pi}{(k\zeta_c)^3}\left(1 + \frac{4}{(m\zeta_c)^2(k\zeta_c)^4}\right)^{-1} \tag{12}$$

It is now straightforward to calculate the static gravitational potential between two unit masses placed on the positive-tension brane at a distance r from each other. This potential is generated by the exchange of the massive modes (cf. Refs. [1,7] – recall that the zero mode here is not localized)

$$V(r) = G_5 \int_0^\infty \, dm \, \frac{e^{-mr}}{r} \, \psi_m^2(z=0) \tag{13}$$

It is convenient to divide this integral into two parts,

$$V(r) = G_5 \int_0^{\zeta_c^{-1}} \, dm \, \frac{e^{-mr}}{r} \, \psi_m^2(0) + G_5 \int_{\zeta_c^{-1}}^\infty \, dm \, \frac{e^{-mr}}{r} \, \psi_m^2(0) \tag{14}$$

At $r \gg k^{-1}$, the second term in Eq. (14) is small and it is similar to the contribution of the continuum modes to the gravitational potential in RS model. It gives short distance corrections to Newton's law,

$$\Delta V_{short}(r) \sim \frac{G_5}{kr^3} = \frac{G_N}{r} \cdot \frac{1}{k^2 r^2} \tag{15}$$

where $G_N = G_5 k$ is the four-dimensional Newton constant.

Of greater interest is the first term in Eq. (14) which dominates at $r \gg k^{-1}$. Substituting the normalization factor (12) into this term, we find

$$V(r) = \frac{G_5}{r} \int_0^{\zeta_c^{-1}} \, dm \frac{\pi}{(k\zeta_c)^3} \left(1 + \frac{4}{(m\zeta_c)^2 (k\zeta_c)^4}\right)^{-1} \frac{4k^2}{\pi^2 m^2} e^{-mr} \tag{16}$$

This integral is always saturated at $m \lesssim r_c^{-1} \ll \zeta_c^{-1}$, where

$$r_c = \zeta_c (k\zeta_c)^2 \equiv k^{-1} e^{3kz_c} \tag{17}$$

Therefore, we can extend the integration to infinity and obtain

$$\begin{aligned} V(r) &= \frac{G_N}{r} \cdot \frac{2}{\pi} \int_0^\infty dx \frac{e^{-\frac{2r}{r_c}x}}{x^2+1} \\ &= \frac{2G_N}{\pi r} \left[\mathrm{ci}(2r/r_c)\sin(2r/r_c) - \mathrm{si}(2r/r_c)\cos(2r/r_c)\right] \end{aligned} \tag{18}$$

where $x = mr_c/2$, and $\mathrm{ci/si}(t) = -\int_t^\infty \frac{\cos/\sin(u)}{u} du$ are the sine and cosine integrals. We see that $V(r)$ behaves in a peculiar way. At $r \ll r_c$, the exponential factor in Eq. (18) can be set equal to one and the four-dimensional Newton law is restored, $V(r) = G_N/r$. Hence, at intermediate distances, $k^{-1} \ll r \ll r_c$, the collection of continuous modes with $m \sim r_c^{-1}$ has the same effect as the graviton bound state in RS model. However, in the opposite case, $r \gg r_c$, we find

$$V(r) = \frac{G_N r_c}{\pi r^2} \tag{19}$$

which has the form of "Newton's law" of five-dimensional gravity with a renormalized gravitational constant.

It is clear from Eq. (18) that at intermediate distances, $k^{-1} \ll r \ll r_c$, the four-dimensional Newtonian potential obtains not only short distance corrections, Eq. (15), but also long distance ones, $V(r) = G_N/r + \Delta V_{short}(r) + \Delta V_{long}(r)$. The long distance corrections are suppressed by r/r_c, the leading term being

$$\Delta V_{long}(r) = \frac{G_N}{r} \cdot \frac{r}{r_c} \cdot \frac{4}{\pi} \left(\ln \frac{2r}{r_c} + \mathbf{C} - 1 \right) \tag{20}$$

where $\mathbf{C}$ is the Euler constant. The two types of corrections, Eqs. (15) and (20), are comparable at roughly $r \sim \zeta_c$. At larger r, deviations from the four-dimensional Newton law are predominantly due to the long-distance effects.

In our scenario, the approximate four-dimensional gravity law is valid over a finite range of distances. Without strong fine-tuning however, this range is large, as required by phenomenology. Indeed, the exponential factor in Eq. (17) leads to a very large r_c even for microscopic separations, z_c, between the branes. As an example, for $k \sim M_{Pl}$ we only require $z_c \sim 50 l_{Pl}$ to have $r_c \sim 10^{28}$ cm, the present horizon size of the Universe, i.e., with mild assumptions about z_c, the four-dimensional description of gravity is valid from the Planck to cosmological scales (in this example, long distance corrections to Newton's gravity law dominate over short distance ones at $r \gtrsim \zeta_c \sim 10^{-13}$ cm).

An interesting consequence of the incomplete localization of graviton is the dissipation into the fifth dimension of gravitational waves propagating to large distances. Let us consider gravitational waves generated by a periodic pointlike source on the brane, $T(x,z) = T(\mathbf{x}) e^{-i\omega t} \delta(z)$, where the four-dimensional indices are again omitted. The gravitational field on the brane is given by the convolution of the source, $T(\mathbf{x})$, with the Green's function

$$G(\mathbf{x} - \mathbf{x}'; \omega) = 8\pi G_5 \int d(t'-t)\ G(x, x'; z = z' = 0) e^{-i\omega(t-t')} \tag{21}$$

Here the five-dimensional gravitational constant is included for convenience of comparison with four-dimensional formulae, and $G(x, x'; z, z')$ is the retarded Green's function of the linearized Einstein equations. The latter is constructed from the full set of eigenmodes in the usual way:

$$G(x, x'; z, z') = \int_0^\infty dm\ \psi_m(z) \psi_m(z') \frac{1}{(2\pi)^4} \int d^4 p \frac{e^{-ipx}}{m^2 - p^2 - i\epsilon p^0} \tag{22}$$

After substitution of (22) into (21) and simplifications we get

$$G(\mathbf{x} - \mathbf{x}'; \omega) = \frac{2G_5}{r} \int_0^\infty dm\ \psi_m^2(0) e^{ip_\omega r} \tag{23}$$

where $r = |\mathbf{x} - \mathbf{x}'|$, $p_\omega = \sqrt{\omega^2 - m^2}$ when $m < \omega$ and $p_\omega = i\sqrt{m^2 - \omega^2}$ when $m > \omega$. We see that the gravitational field on the brane has the form of a superposition of massive four-dimensional modes. Only modes with $m < \omega$ are actually radiated, the other ones exponentially fall off from the source. Thus, as long as we are interested in gravitational waves, we can integrate in (23) only up to $m = \omega$.

Let us study the case $r_c^{-1} \ll \omega \ll k$. Then, the main contribution to (23) is given by modes with $m \sim r_c^{-1}$, whereas modes with larger masses give rise to corrections suppressed

by ω/k (cf.(14), (15)). In this region of m, the eigenfunctions ψ_m are given by the explicit expressions (7), (12) and p_ω is approximated by $\omega - \frac{m^2}{2\omega}$. In this way we get (cf.(18))

$$G(\mathbf{x}-\mathbf{x}';\omega) = \frac{2G_N}{r} e^{i\omega r} \cdot \frac{2}{\pi} \int_0^\infty dx \frac{e^{-i\frac{2r}{\omega r_c^2}x^2}}{1+x^2} \tag{24}$$

where again $x = mr_c/2$ and we extended the integration to infinity. This integral is expressed in terms of the error function:

$$G(\mathbf{x}-\mathbf{x}';\omega) = \frac{2G_N}{r} e^{i\omega r} \left(1 - \mathrm{erf}\,(\beta)\right) e^{\beta^2} \tag{25}$$

where $\beta = e^{i\frac{\pi}{4}}\sqrt{\frac{2r}{\omega r_c^2}}$. When $\beta \ll 1$ (that is $r \ll \omega r_c^2$) one has $\mathrm{erf}\,(\beta) \approx 0$, and we obtain the usual $1/r$-dependence of the gravity wave amplitude on the distance to the source. In the opposite case $\beta \gg 1$ ($r \gg \omega r_c^2$), we make use of $\mathrm{erf}\,(\beta) = 1 - \frac{1}{\sqrt{\pi}\beta} e^{-\beta^2}$ to obtain

$$G(\mathbf{x}-\mathbf{x}';\omega) = \frac{2G_N}{r^{3/2}} \sqrt{\frac{\omega r_c^2}{2\pi}} e^{i\omega r - i\frac{\pi}{4}} \tag{26}$$

that is, the amplitude is proportional to $r^{-3/2}$. Thus, the gravitational waves dissipate into the fifth dimension and from the point of view of a four-dimensional observer on the brane, energy of gravity waves is not conserved.

This effect becomes considerable after the wave travels the distance of order $r \sim \omega r_c^2$ from the source. Note that at $\omega \gg r_c^{-1}$ this distance is much larger than the distance r_c at which the violation of four-dimensional Newton's law is appreciable. This difference between the two distance scales can in fact be seen to be a relativistic effect. The collection of five-dimensional graviton states with $m \sim r_c^{-1}$ may be viewed as an RS bound state which becomes metastable in our model. The width of this metastable state is of order $\Gamma = \Delta m \sim m \sim r_c^{-1}$. In its own reference frame, the graviton disappears into the fifth dimension with time scale $\tau \sim \Gamma^{-1} \sim r_c$. This time scale determines the distance at which four-dimensional gravity of *static* sources gets modified. On the other hand, when the graviton moves in four dimensional space-time with momentum $p \sim \omega$, there is an additional gamma-factor of order $\gamma \sim \omega/m \sim \omega r_c$. The graviton therefore remains effectively four-dimensional during a time of order $\gamma\tau \sim \omega r_c^2$ due to the relativistic time delay.

Because of this property, the leaking of the gravity waves into the extra dimension is negligible at relatively short wavelengths. However, the corresponding time scale becomes of order r_c for wavelengths of the same order. This is another manifestation of our observation that extra dimensions open up at the length scale r_c.

Finally, we point out that in more complicated higher-dimensional models, the long distance properties of gravitational interactions may be even more intriguing. Indeed, let us modify the model discussed throughout this paper by compactifying the fifth dimension to a very large radius, $z_* \gg z_c$. Then the mass spectrum of Kaluza–Klein gravitons becomes discrete, and the graviton zero mode reappears. However, the spacing between the masses may be tiny, depending on z_*. With appropriately chosen z_*, the five-dimensional gravity law, Eq. (19), will itself be valid in a finite interval of distances, and the four-dimensional gravity will again be restored at largest scales, well above r_c.

We conclude that higher-dimensional theories provide, somewhat unexpectedly, valid alternatives to four-dimensional gravity at *large distances*. With hindsight, it is perhaps not surprising that this happens, since at the very large scale, the anti-de Sitter sandwich becomes very slim, and spacetime is nearly flat, however, from the perspective of our putative universe – the four-dimensional central wall – this conclusion is not so transparent. It would be worthwhile to explore such models in the context of cosmology and astrophysics. The long-distance phenomena in our Universe may become a window to microscopic extra dimensions!

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